CUSTOM STIFFNESS MATRIX IN AXISVM

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User's guide to domains with custom stiffness matrix in AxisVM

This guide is entitled to give a deeper insight to the theoretical background behind the hood of AxisVM, helping the users in the process of arriving to a state of better understanding. The chapters cover the most popular plate theories and worked out examples.

The document doesn't lead to full enlightenment in the topic and the theory covered here should only be regarded as a brief introduction to plate theory and in fact is just a compendious summary of the outstanding lecture note of **Carlos A. Felippa** [2]. We mention here, that some of the figures praise his work, as it is noted in the captions. For further information about plate theory and a broad literature review the reader is directed to the original source of this paper:
https://www.colorado.edu/engineering/CAS/courses.d/AFEM.d/AFEM.Ch20.d/AFEM.Ch20.pdf

1 Theoretical background

The brief theoretical background is laid down in this section, providing the foundation for the case when one needs to develop expressions without literature suggestion. The most important statements are printed with bold letters, so if one is only interested in the practical application of custom stiffness matrices in AxisVM, they can run through this section without missing the point.

1.1 Field Equations of Flat Shells

In this chapter we will study plates in a plane stress state, also called a membrane or lamina state in the literature. This state occurs if the external loads act on the plate midsurface as sketched in the following Figure.

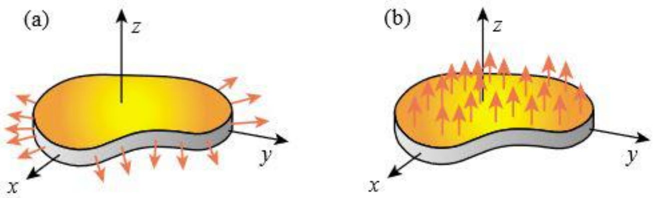

Figure 1: A flat shell structure in: (a) plane stress, (b) bending state. (Carlos A. Felippa)

Under these conditions the distribution of stresses and strains across the thickness may be viewed as uniform, and the 3D problem can easily be reduced to 2D. If the structure shows linear elastic behavior under the action of applied loads, than we have effectively reduced the problem to that of two-dimensional elasticity.

1.1.1 Kinematic Equations

Consider a 3D body having an arbitrary shape in the x-y plane, and bounded by surfaces $z = t_{top}$ and $z = t_{bottom}$, so for every point of the 3D body $t_{top} \geq z \geq t_{bottom}$. For the sake of simplicity, from now on we will use the notation $t_t = t_{top}$ and $t_b = t_{bottom}$. We define the functions $u = u(x, y, z)$, $v = v(x, y, z)$ and $z = z(x, y, z)$ to describe the displacement of a point in the direction of the axes x, y and z respectively. Our intention is to approximate this displacement field by selecting kinematic variables on a reference surface, defined by the points $z = 0$. That is called the midsurface of the plate. If we assume small displacements, and that plane sections remain plane after deformation, we can say that the displacement of a point $P(x_p, y_p, z_p)$ can be described by a rigid body motion of the point $(x_p, y_p, 0)$ and two rotations $\theta_x(x_p, y_p)$ and $\theta_y(x_p, y_p)$. The aforementioned rotations are considered to be positive according to Figure 1.

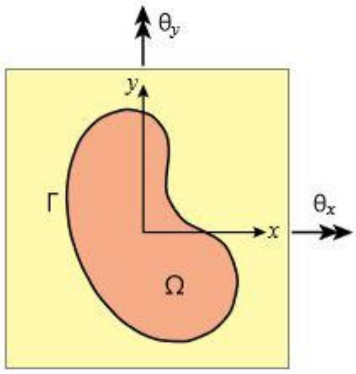

Figure 2: Positive rotations. (Carlos A. Felippa)

According to the assumptions so far, the approximated displacements of a general plate particle $P(x, y, z)$ are given by

$$\begin{aligned} u(x, y, u) &\cong u_0(x, y) + z\theta_y(x, y) \\ v(x, y, u) &\cong v_0(x, y) - z\theta_x(x, y) \\ w(x, y, u) &\cong w_0(x, y), \end{aligned} \tag{1}$$

where zero index here and from now on refer to values being measured on the midsurface. Through the derivations we also simplify our notation by generally writing f instead of $f(x_1, x_2, \ldots x_n)$, when the arguments are trivial. The small engineering strains associated with these equations are obtained from well known elasticity equations:

$$\begin{aligned} \varepsilon_x &= \frac{\partial u}{\partial x} = \frac{\partial u_0}{\partial x} + z\frac{\partial \theta_y}{\partial x} = \varepsilon_{0_x} + z\kappa_x \\ \varepsilon_y &= \frac{\partial v}{\partial y} = \frac{\partial v_0}{\partial y} - z\frac{\partial \theta_x}{\partial y} = \varepsilon_{0_y} + z\kappa_y \\ \gamma_{xy} &= \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = \frac{\partial u_0}{\partial y} + \frac{\partial v_0}{\partial x} + z(\frac{\partial \theta_y}{\partial y} - \frac{\partial \theta_x}{\partial x}) = \gamma_{0_{xy}} + z\kappa_{xy} \\ \gamma_{xz} &= \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} = \theta_y + \frac{\partial w_0}{\partial x} \\ \gamma_{yz} &= \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} = -\theta_x + \frac{\partial w_0}{\partial y} \end{aligned} \tag{2}$$

The listed nonzero strains are usually grouped to having in-plane and out-of-plane strains. For the in-plane strains, the following notation is popular:

$$\boldsymbol{\varepsilon} = \begin{pmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{pmatrix} = \begin{pmatrix} \varepsilon_{0_x} \\ \varepsilon_{0_y} \\ \gamma_{0_{xy}} \end{pmatrix} + z \begin{pmatrix} \kappa_x \\ \kappa_y \\ \kappa_{xy} \end{pmatrix} = \boldsymbol{\varepsilon}_0 + z\boldsymbol{\kappa}. \tag{3}$$

1.1.2 Physical and Equilibrium Equations

Each of the introduced nonzero strains induce a corresponding work conjugate stress pair, namely $\sigma_x, \sigma_y, \tau_{xy}, \tau_{xz}$ and τ_{yz}. Continuing the previous thoughts and further assuming that the plate is isotropic and homogeneous, the stresses and strains are related by Hooke's law for plane stress at every material point. In matrix form this relationship is

$$\boldsymbol{\sigma} = \begin{pmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{pmatrix} = \frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix} \boldsymbol{\varepsilon} = \mathbf{Q}_m \boldsymbol{\varepsilon} = \mathbf{Q}_m \boldsymbol{\varepsilon}_0 + z\mathbf{Q}_m \boldsymbol{\kappa} \tag{4}$$

for the in-plane stresses and

$$\boldsymbol{\tau} = \begin{pmatrix} \tau_{xz} \\ \tau_{yz} \end{pmatrix} = \begin{bmatrix} G & 0 \\ 0 & G \end{bmatrix} \begin{pmatrix} \gamma_{xz} \\ \gamma_{yz} \end{pmatrix} = \mathbf{Q}_s \boldsymbol{\gamma}, \tag{5}$$

for the out-of-plane stresses, where G stands for the shear modulus. We formulate equations (4) and (5) with a notation similar to the so-called Kelvin-Voight notation, which kind of standardizes mapping the notation of stress and strain related parameters. The only difference is that we interchanged the notations for τ_{xz} and τ_{yz} and finally applied the following definitions:

$$(\sigma_x, \sigma_y, \sigma_z, \tau_{xz}, \tau_{yz}, \tau_{xy}) \equiv (\sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5, \sigma_6), \tag{6}$$

and therefore we have

$$\begin{pmatrix} \sigma_1 \\ \sigma_2 \\ \tau_6 \end{pmatrix} = \begin{bmatrix} Q_{11} & Q_{12} & Q_{16} \\ Q_{12} & Q_{22} & Q_{26} \\ Q_{16} & Q_{26} & Q_{66} \end{bmatrix} \begin{pmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_6 \end{pmatrix} \tag{7}$$

and

$$\begin{pmatrix} \sigma_4 \\ \sigma_5 \end{pmatrix} = \begin{bmatrix} Q_{44} & Q_{45} \\ Q_{45} & Q_{55} \end{bmatrix} \begin{pmatrix} \varepsilon_4 \\ \varepsilon_5 \end{pmatrix}. \tag{8}$$

It is natural that the corresponding work-conjugate strain pairs of the stresses listed in (6) inherit the same indexing scheme.

Stress resultants can be calculated by integrating the elementary stresses and stress couples with respect to z, according to an arbitrary sign convention. The used sign conventions for moments and shear forces are depicted in Figure 2. The sign conventions for in-plane forces are inherited from the definitions of positive stresses and therefore are straightforward. With this selection the stress resultants are:

$$\begin{aligned}
\mathbf{N} &= \begin{pmatrix} N_x \\ N_y \\ N_{xy} \end{pmatrix} = \int_{t_b}^{t_t} \boldsymbol{\sigma} dz = (t_t - t_b)\mathbf{Q}_m \boldsymbol{\varepsilon}_0 + \frac{1}{2}(t_t^2 - t_b^2)\mathbf{Q}_m \boldsymbol{\kappa} \\
&= \mathbf{A}\boldsymbol{\varepsilon}_0 + \mathbf{B}\boldsymbol{\kappa}, \\
\mathbf{M} &= \begin{pmatrix} M_x \\ M_y \\ M_{xy} \end{pmatrix} = \int_{t_b}^{t_t} z\boldsymbol{\sigma} dz = \frac{1}{2}(t_t^2 - t_b^2)\mathbf{Q}_m \boldsymbol{\varepsilon}_0 + \frac{1}{3}(t_t^3 - t_b^3)\mathbf{Q}_m \boldsymbol{\kappa} \\
&= \mathbf{B}\boldsymbol{\varepsilon}_0 + \mathbf{D}\boldsymbol{\kappa} \\
\mathbf{Q} &= \begin{pmatrix} Q_x \\ Q_y \end{pmatrix} = \int_{t_b}^{t_{top}} \boldsymbol{\tau} dz = (t_t - t_b)\mathbf{Q}_s \boldsymbol{\gamma} = \mathbf{S}\boldsymbol{\gamma}.
\end{aligned} \tag{9}$$

The equations of (9) can be encompassed in the following form:

$$\begin{pmatrix} \mathbf{N} \\ \mathbf{M} \end{pmatrix} = \begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{B} & \mathbf{D} \end{bmatrix} \begin{pmatrix} \boldsymbol{\varepsilon}_0 \\ \boldsymbol{\kappa} \end{pmatrix}, \quad \mathbf{Q} = \mathbf{S}\boldsymbol{\gamma}, \tag{10}$$

or

$$\begin{pmatrix} \mathbf{N} \\ \mathbf{M} \\ \mathbf{Q} \end{pmatrix} = \begin{bmatrix} \mathbf{A} & \mathbf{B} & \\ \mathbf{B} & \mathbf{D} & \\ & & \mathbf{S} \end{bmatrix} \begin{pmatrix} \boldsymbol{\varepsilon}_0 \\ \boldsymbol{\kappa} \\ \boldsymbol{\gamma} \end{pmatrix}. \tag{11}$$

In equation (11) the sub-matrices $\mathbf{A}$ and $\mathbf{D}$ characterizes the membrane and flexural stiffness of a plate element, while sub-matrix $\mathbf{B}$ represents a coupling between flexural and membrane effects. That is, for example if $\mathbf{B}$ is not empty, and the plate undergoes in plane deformation only ($\boldsymbol{\kappa} = \mathbf{0}$), moments will be present and the same coupling applies vice versa. It is clear to see, that $\mathbf{B}$ is a null matrix if and only if $|t_{bottom}| = |t_{top}|$. In other words we can say, that for a shell of constant thickness, $\mathbf{B}$ represents the eccentric effects. Sub matrix $\mathbf{S}$ contains the stiffness terms against transverse shearing.

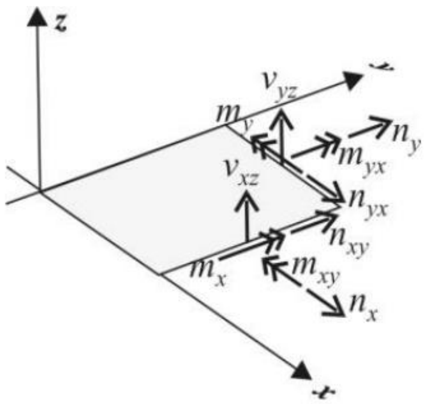

Figure 3: Sign convention for bending moments.

To derive interior equilibrium equations we consider differential midsurface elements $dx \times dy$ aligned with the x, y axes as illustrated in Figure 4.

Without the technical details of derivation, the equilibrium equations are given by

$$\begin{aligned}
\Sigma F_x &: \frac{\partial N_x}{\partial x} + \frac{\partial N_{xy}}{\partial y} + q_x &= 0 \\
\Sigma F_y &: \frac{\partial N_{yx}}{\partial x} + \frac{\partial N_y}{\partial y} + q_y &= 0 \\
\Sigma F_z &: \frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} + q_z &= 0 \\
\Sigma M_x &: \frac{\partial M_{yx}}{\partial x} + \frac{\partial M_y}{\partial y} - Q_y &= 0 \\
\Sigma M_y &: \frac{\partial M_x}{\partial x} + \frac{\partial M_{xy}}{\partial y} - Q_x &= 0 \\
\Sigma M_z &: M_{xy} - M_{yx} &= 0.
\end{aligned} \tag{12}$$

1.2 Shear correction

For a homogeneous plate and using an equilibrium argument similar to Euler-Bernoulli beams, the stresses γ_{xz} and γ_{yz} can be shown to have a parabolic distribution over the thickness. This is inconsistent with the postulated kinematic relations, which says that the shear strains, therefore the shear stresses are constant through the thickness. Thus, the shear stiffness values must be corrected. Since the originally three dimensional body is now represented with a single surface, all the points along a material section are represented by the point of intersection of the material section and the reference plane $z=0$. If the stress values of that one point don't fit a realistic distribution, it can lead to an over- or underestimation of the strain energy density at that point.

Here the two stage calculation of the shear correction factor for a pate under discussion is illustrated for τ_{xz} at a fixed point on the midsurface (x_p, y_p). First let assume, that it shows a parabolic distribution through the thickness by having $\tau_{xz}(z)=\tau_{xz}(x_p, y_p, z)$ in the following form:

$$\tau_{xz}(z) = p_0 + p_1 z + p_2 z^2. \tag{13}$$

Since basic equilibrium considerations dictate the transverse shear stress distribution to be symmetric with respect to the midsurface of the plate, to vanish on the bottom and top surfaces of the plate and to have a peak value on the midsurface. Also, the shear stress function integrated through the thickness must be equivalent to the shear force Q_x. Altogether, the assumed function has to satisfy the following criteria:

$$\begin{aligned} \tau_{xz}(t_t) &= 0, \\ \tau_{xz}(t_b) &= 0, \\ \frac{\partial \tau_{xz}(z)}{\partial z}\Big|_{z=(t_t+t_b)/2} &= 0, \\ \int_{t_b}^{t_t} \tau_{xz}(z)dz &= Q_x. \end{aligned} \tag{14}$$

Doing the math one can arrive to a form of

$$\tau_{xz}(z) = Q_x(C_0 + C_1 z + C_2 z^2), \tag{15}$$

with

$$\begin{aligned} C_2 &= \left(\frac{1}{3}\left(t_t^3 - t_b^3\right) - \frac{1}{2}\left(t_t^2 - t_b^2\right)(t_b + t_t) + t_b t_t (t_t - t_b)\right)^{-1}, \\ C_1 &= -C_2(t_t + t_b) \\ C_0 &= C_2 t_t t_b. \end{aligned} \tag{16}$$

The second step is to find a correction ξ_x to the shear stiffness terms so, that the shear strain density calculated from the constant distribution Π_c and from the parabolic one Π_p coalesce. Utilizing that $\gamma_{xz}(z) = \tau_{xz}(z)/G$ in the case of the parabolic distribution and $\gamma_{xz} = \tau_{xz}/(\xi_x G)$ for the constant one at every material point, we can write

$$\begin{aligned} 2\Pi_p &= \int_{t_b}^{t_t} \tau_{xz}(z)\gamma_{xz}(z)dz = \int_{t_b}^{t_t} \frac{\tau_{xz}(z)^2}{G}dz \\ 2\Pi_c &= \int_{t_b}^{t_t} \tau_{xz}\gamma_{xz}dz = \int_{t_b}^{t_t} \frac{\tau_{xz}^2}{\xi_x G}dz. \end{aligned} \tag{17}$$

From the condition $\Pi_c = \Pi_p$, we can arrive to the value of the shear correction factor for the x direction to be $\xi_x = 5/6$. The same derivation can be carried out for the y direction, producing a value of $\xi_y = 5/6$. With this modification, equation (5) takes the form of

$$\boldsymbol{\tau} = \begin{pmatrix} \tau_{xz} \\ \tau_{yz} \end{pmatrix} = \begin{bmatrix} \xi_x G & 0 \\ 0 & \xi_y G \end{bmatrix} \begin{pmatrix} \gamma_{xz} \\ \gamma_{yz} \end{pmatrix} = \mathbf{Q}_s^* \boldsymbol{\gamma}., \tag{18}$$

Consequently we have

$$\begin{pmatrix} Q_x \\ Q_y \end{pmatrix} = (t_t - t_b)\mathbf{Q}_s^* \boldsymbol{\gamma} = \mathbf{S}^* \boldsymbol{\gamma}, \tag{19}$$

and the ABD matrix of a Mindlin-Reissner plate of a material point on the midsurface takes it's final form as

$$\begin{pmatrix} \mathbf{N} \\ \mathbf{M} \\ \mathbf{Q} \end{pmatrix} = \begin{bmatrix} \mathbf{A} & \mathbf{B} & \\ \mathbf{B} & \mathbf{D} & \\ & & \mathbf{S}^* \end{bmatrix} \begin{pmatrix} \boldsymbol{\varepsilon}_0 \\ \boldsymbol{\kappa} \\ \boldsymbol{\gamma} \end{pmatrix}, \tag{20}$$

And in more detail

$$\begin{pmatrix} Nx \\ Ny \\ Nxy \\ Mx \\ My \\ Mxy \\ Qx \\ Qy \end{pmatrix} = \begin{bmatrix} \begin{bmatrix} A_{11} & A_{12} & A_{16} \\ A_{12} & A_{22} & A_{26} \\ A_{16} & A_{26} & A_{66} \end{bmatrix} & \begin{bmatrix} B_{11} & B_{12} & B_{16} \\ B_{12} & B_{22} & B_{26} \\ B_{16} & B_{26} & B_{66} \end{bmatrix} 3mm & \\ \begin{bmatrix} B_{11} & B_{12} & B_{16} \\ B_{12} & B_{22} & B_{26} \\ B_{16} & B_{26} & B_{66} \end{bmatrix} & \begin{bmatrix} D_{11} & D_{12} & D_{16} \\ D_{12} & D_{22} & D_{26} \\ D_{16} & D_{26} & D_{66} \end{bmatrix} 3mm & \\ & & \begin{bmatrix} S_{44}^* & S_{45}^* \\ S_{45}^* & S_{55}^* \end{bmatrix} \end{bmatrix} \begin{pmatrix} \varepsilon_{0_x} \\ \varepsilon_{0_y} \\ \gamma_{0_{xy}} \\ \kappa_x \\ \kappa_y \\ \kappa_{xy} \\ \gamma_{xz} \\ \gamma_{yz} \end{pmatrix}, \tag{21}$$

where for a homogeneous plate of constant thickness

$$\begin{aligned} A_{ij} &= (t_t - t_b) Q_{m_{ij}}, \\ B_{ij} &= \frac{1}{2}(t_t^2 - t_b^2) Q_{m_{ij}}, \\ D_{ij} &= \frac{1}{3}(t_t^3 - t_b^3) Q_{m_{ij}}, \end{aligned} \tag{22}$$

and $S_{ij}*$ are the shear corrected stiffness values. **The form and the meaning of the parameters A_{ij}, B_{ij}, D_{ij} and S_{ij}^* of the material equations for a plate section in equation (21) is exactly the same that is shown in AxisVM**, illustrated in Figure 3.

$$\begin{bmatrix} n_x \\ n_y \\ n_{xy} \\ m_x \\ m_y \\ m_{xy} \\ v_{xz} \\ v_{yz} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} & A_{16} & B_{11} & B_{12} & B_{16} & & \\ A_{12} & A_{22} & A_{26} & B_{12} & B_{22} & B_{26} & \multicolumn{2}{c}{0} \\ A_{16} & A_{26} & A_{66} & B_{16} & B_{26} & B_{66} & & \\ B_{11} & B_{12} & B_{16} & D_{11} & D_{12} & D_{16} & & \\ B_{12} & B_{22} & B_{26} & D_{12} & D_{22} & D_{26} & \multicolumn{2}{c}{0} \\ B_{16} & B_{26} & B_{66} & D_{16} & D_{26} & D_{66} & & \\ & 0 & & & 0 & & S_{44} & S_{45} \\ & & & & & & S_{45} & S_{55} \end{bmatrix} \cdot \begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \\ \kappa_x \\ \kappa_y \\ \kappa_{xy} \\ \gamma_{xz} \\ \gamma_{yz} \end{bmatrix}$$

Figure 4: The ABD matrix in AxisVM, with the corrected shear stiffnesses.

We note that in the case of non-homogeneous plates, the essential steps of the calculation of shear correction factors are the same, but the integration over the plate thickness can be significantly more laborious.

1.3 Kirchhoff-Love Shells

The theory is also known as the "classical plate theory" or "theory of thin plates", as it **accounts for no shear deformation**, and is applicable when the thickness of the plate is not so thin that the lateral deflection of the plate becomes comparable to it.

The kinematics of the Kirchhoff-Love plate is based on the extensions of Euler-Bernoulli beam theory to the case of biaxial bending. Next to the previously stated assumptions, this means that **material normals to the midsurface remain normal to the deformed reference surface**, see Figure 23. This assumption relates the rotations to the slopes:

$$\theta_x = \frac{\partial w}{\partial y}, \quad \theta_y = -\frac{\partial w}{\partial x}. \tag{23}$$

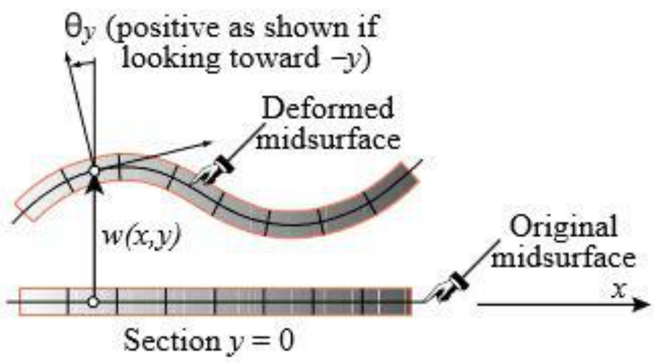

Figure 5: The normality assumption of a Kirchhoff-Love plate. (Carlos A. Felippa)

If we substitute the relations of equations (23) into equation (2), the following is obtained

$$\begin{aligned} \varepsilon_x &= \frac{\partial u_0}{\partial x} - z\frac{\partial^2 w}{\partial x^2} = \varepsilon_{0_x} + z\kappa_x \\ \varepsilon_y &= \frac{\partial v_0}{\partial y} - z\frac{\partial^2 w}{\partial y^2} = \varepsilon_{0_y} + z\kappa_y \\ \gamma_{xy} &= \frac{\partial u_0}{\partial y} + \frac{\partial v_0}{\partial x} - 2z\frac{\partial^2 w}{\partial x \partial y} = \gamma_{0_{xy}} + 2z\kappa_{xy} \\ \gamma_{xz} &= -\frac{\partial w}{\partial x} + \frac{\partial w}{\partial x} = 0 \\ \gamma_{yz} &= -\frac{\partial w}{\partial y} + \frac{\partial w}{\partial y} = 0 \end{aligned} \tag{24}$$

Remarks on the Kirchhoff-Love theory: It is important to note, that some **inconsistencies of Kirchhoff-Love theory** emerge on taking a closer look at equations (24). For example, the transverse shear strains are zero, which immediately implies $\tau_{xz} = \tau_{yz} = 0$ and consequently there are no shear forces, so $Q_x = Q_y = 0$. But these forces appear necessarily from the equilibrium equations (12). Similarly, $\varepsilon_z = 0$ says that the plate is in a state of plane strain, whereas a plane stress ($\sigma_z = 0$) is closer to the physical reality. For a homogeneous isotropic plate, **plane strain and plain stress coalesce if and only if the Poisson's ratio is zero**. These inconsistencies have been the topic of hundreds of learned papers that fill applied mechanics journals but nobody reads. Furthermore, since the theory predicts zero transverse shear stress, for it to be an acceptable approximation, these stresses should be significantly smaller than the maximum in-plane stresses. This is the case if the thickness of the plate is thin (but not very thin). If these conditions are not met, one should move to the Mindlin-Reissner model, which accounts for transverse shear energy to first order.

However, the theory has some practical benefits. In the absence of shear forces one doesn't need to determine the entries of sub-matrix $\mathbf{S}$ in equations (11) and (10). The necessary values for the shear forces are only dictated by the equilibrium equations and their calculation is shifted to the post-processing stage.

As a final note, if we consider the case, when $|t_b| = |t_t|$, rearrange the moment-equilibrium equations around x and y in equations (12) for Q_x and Q_y, substitute the moment-curvature relations from equation (10) and use the definitions for curvatures from equation (3), we come to the famous biharmonic equation, first derived by Lagrange in 1813:

$$D_{11}\frac{\partial^4 w}{\partial x^4} + 2(D_{12} + 2D_{66})\frac{\partial^4 w}{\partial x^2 \partial y^2} + D_{22}\frac{\partial^4 w}{\partial y^4} = p_z, \tag{25}$$

or with the notation of $H = D_{12} + 2D_{66}$,

$$D_{11}\frac{\partial^4 w}{\partial x^4} + 2H\frac{\partial^4 w}{\partial x^2 \partial y^2} + D_{22}\frac{\partial^4 w}{\partial y^4} = p_z, \tag{26}$$

whereas the same equation in the famous book of Timoschenko [3], with different notations of the rigidities,

$$D_x\frac{\partial^4 w}{\partial x^4} + 2H\frac{\partial^4 w}{\partial x^2 \partial y^2} + D_y\frac{\partial^4 w}{\partial y^4} = p_z, \tag{27}$$

with $H = D_1 + 2D_{xy}$.

1.4 Mindlin-Reissner Shells

The theory is also known as the "first order shear deformation theory" or "theory of thin and moderately thick plates", as it accounts for shear deformation to the first order, and is applicable when the thickness of the plate is thin or moderately thick. The core difference in the formulation when compared to the Kirchhoff-Love theory is that the normality condition is relaxed, thus the rotations θ_x and θ_y and therefore the strain state are not defined by the deflection function w only. In this case the strain-displacement equations under in equations (2) apply. As a consequence, the shear strains γ_{xz} and γ_{yz} are not zero, nor the shear stresses τ_{xz} and τ_{yz}. **This lead to the fact, that the values of sub-matrix $\mathbf{S}$ in equation (11) in addition with the proper values of the shear correction factors should also be properly determined.** It is also worth mentioning, that the Mindlin-Reissner theory can account for thin plates too. **If the shell is thin, it's shear rigidity is very high, in fact infinitely high as Kirchhoff's assumptions suggest. Therefore in that case, if the user has no better idea, the corrected shear rigidity in both directions (S_{44}^* and S_{55}^*) can be set to infinity, so producing the same result, as if the calculations were carried out on the basis of the Kirchoff-Love theory. If the shell is not thin, the calculation of shear correction factors in both directions is inevitable.**

2 Examples

In AxisVM it is assumed that the usage of a domain with a custom ABD matrix is motivated by the the users being able to provide the necessary stiffness values by themselves or from the literature.

2.1 Example 1 : Bending of a corrugated sheet

This example is worked out on the basis of the suggestions to the bending rigidities of corrugated sheets in [3, p.~367].

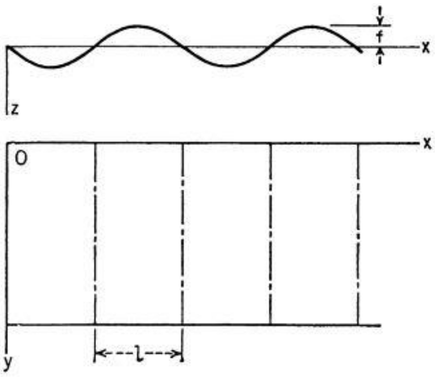

Figure 6: Parameters of a corrugated sheet. (Timoshenko)

Let say we want our domain to have the stiffness of a corrugated sheet shown in Figure 5. Assume that we have a plate model, so there is no need for membrane effects. In this case we don't have to determine the values A_{ij}, and the corresponding entries on the AxisVM panel will be inactive. Let E and ν be the elastic constants of the material of the sheet, h its thickness,

$$z = f \sin \frac{\pi x}{l} \tag{28}$$

the form of the corrugation and s the length of the arc of one-half a wave. According to the suggestions for the plate rigidities:

$$\begin{aligned} D_{11} &= \frac{l}{s} \frac{Eh^3}{12(1-\nu^2)} \\ D_{22} &= EI \\ D_{12} &\cong 0 \\ H &= 2D_{66} = \frac{l}{s} \frac{Eh^3}{12(1+\nu)}, \end{aligned} \tag{29}$$

in which approximately

$$\begin{aligned} s &= l(1+\frac{\pi^2 f^2}{4l^2}), \\ I &= \frac{f^2 h}{2}[1-\frac{0.81}{1+2.5(\frac{f}{2l})^2}]. \end{aligned} \tag{30}$$

For example, if we take $E = 21000 kN/cm^2$, $\nu = 0.3$, $l = 50cm$, $h = 1cm$ and $f = 5cm$, matrix D in equation (21) and in Figure 3 equals to

$$\mathbf{D} = \begin{bmatrix} 1876.76 & 0. & 0. \\ 0. & 51195.65 & 0. \\ 0. & 0. & 656.86 \\ & & \end{bmatrix}. \tag{31}$$

Unfortunately, the source does not provide suggestions for the shear stiffness values S_{ij}. However, an acceptable estimation would be to have (for this example) $S_{44}{}^{*} = S_{55}{}^{*} = \frac{5}{6}Gh$, where $G = \frac{E}{2(1+\nu)}$ is the shear modulus. With this setting, the stiffness matrix of the plate element under discussion would be

$$\begin{pmatrix} \mathbf{D} \\ \mathbf{S} \end{pmatrix} = \begin{bmatrix} 1876.76 & 0. & 0. & 0. & 0. \\ 0. & 51195.65 & 0. & 0. & 0. \\ 0. & 0. & 656.86 & 0. & 0. \\ 0. & 0. & 0. & 6730.76 & 0. \\ 0. & 0. & 0. & 0. & 6730.76 \\ & & & & \end{bmatrix}. \tag{32}$$

2.2 Example 2 : Bending of a voided plate

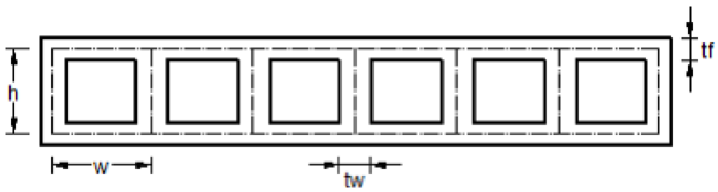

Figure 7: Parameters of a voided plate.

Investigate a voided plate and let the voids be parallel with the local y coordinate axis of the domain. Let E and ν be the elastic constants of the material. According to the suggestions of Basu and Dawson [1], the rigidities of a voided plate with geometrical parameters illustrated in Figure 6 are:

$$\begin{aligned} D_{11} &= \frac{Et_f h^2}{2(1-\nu^2)}, \\ D_{22} &= D_{11}(1+\frac{t_w h}{t_f w}), \\ D_{12} &= \nu D_{11}, \\ D_{66} &= \frac{Gt_f h^2}{2}, \\ S_{44} &= \frac{2Et_f{}^3}{w^2(1+2(h/w)(t_f/t_w)^3)}, \\ S_{55} &= Gt_f h(1+t_f/h)/(t_f w/t_w), \end{aligned} \tag{33}$$

with G being the shear modulus. With the setting of $E = 2860kN/cm^2$, $\nu = 0.2$, $w = 10cm$, $t_w = 5cm$, $t_f = 4cm$, $h = 15cm$, and assuming a shear correction factor of $5/6$ we have the following approximation:

$$\begin{pmatrix} \mathbf{D} \\ \mathbf{S} \end{pmatrix} = \begin{bmatrix} 1340625. & 268125. & 0. & 0. & 0. \\ 268125. & 3854296.8 & 0. & 0. & 0. \\ 0. & 0. & 536250. & 0. & 0. \\ 0. & 0. & 0. & 1202.9 & 0. \\ 0. & 0. & 0. & 0. & 9434. \end{bmatrix}. \tag{34}$$

References

[1] AKj Basu and JM Dawson. Orthotropic sandwich plates. In *Inst Civil Engineers Proc, London/UK/*, 1970.

[2] Carlos A. Felippa. Kirchhoff plates : Field equations. **https://www.colorado.edu/engineering/CAS/courses.d/AFEM.d/AFEM.Ch20.d/AFEM.Ch20.pdf**. (Accessed on 01/30/2018).

[3] S. Timoshenko and S. Woinowsky-Krieger. *Theory of plates and shells*. Engineering societies monographs. McGraw-Hill, 1959.